

# Explicit Simplicial Discretization of Distributed-Parameter Port-Hamiltonian Systems

Marko Seslija<sup>a</sup>, Jacquélien M.A. Scherpen<sup>a</sup>, Arjan van der Schaft<sup>b</sup>

<sup>a</sup>*Department of Discrete Technology and Production Automation, Faculty of Mathematics and Natural Sciences, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands*

<sup>b</sup>*Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands*

---

## Abstract

Simplicial Dirac structures as finite analogues of the canonical Stokes-Dirac structure, capturing the topological laws of the system, are defined on simplicial manifolds in terms of primal and dual cochains related by the coboundary operators. These finite-dimensional Dirac structures offer a framework for the formulation of standard input-output finite-dimensional port-Hamiltonian systems that emulate the behavior of distributed-parameter port-Hamiltonian systems. This paper elaborates on the matrix representations of simplicial Dirac structures and the resulting port-Hamiltonian systems on simplicial manifolds.

*Key words:* Port-Hamiltonian systems, Dirac structures, distributed-parameter systems, structure-preserving discretization, discrete geometry

---

## 1 Introduction

A wide class of field theories can be treated as port-Hamiltonian systems (van der Schaft & Maschke, 2002), (M. Schöberl & K. Schlacher, 2011). The Stokes-Dirac structure defined in (van der Schaft & Maschke, 2002) is an infinite-dimensional Dirac structure which provides a theoretical account that permits the inclusion of varying boundary variables in the boundary problem for partial differential equations. From an interconnection and control viewpoint, such a treatment of boundary conditions is essential for the incorporation of energy exchange through the boundary, since in many applications the interconnection with the environment takes place precisely through the boundary. For numerical integration, simulation and control synthesis, it is of paramount interest to have finite-dimensional approximations that can be interconnected to one another.

Most of the numerical techniques emanating from the field of numerical analysis, however, fail to capture the intrinsic system structures and properties, such as symplecticity, conservation of momenta and energy, as well as differential gauge symmetry. Mixed finite element methods can be constructed in a such a manner that a number of important structural properties are preserved (Bossavit, 1998), (Hirani, 2003), (Hiptmair, 2002). Most of the efforts have been focused on systems on manifolds without boundary or zero energy flow through the boundary. In (Golo, Talasila, van der Schaft, & Maschke, 2004) a mixed finite element

scheme for structure-preserving discretization of port-Hamiltonian systems was proposed. The construction is clear in a one-dimensional spatial domain, but becomes complicated for higher spatial domains. Furthermore, the geometric content of the discretized variables remains moot, in sense that, for instance, the boundary variables do not genuinely live on the geometric boundary.

Recently in (Seslija, van der Schaft, & Scherpen, 2011), we have suggested a discrete exterior geometry approach to structure-preserving discretization of distributed-parameter port-Hamiltonian systems. The spatial domain in the continuous theory represented by a finite-dimensional smooth manifold is replaced by a homological manifold-like simplicial complex and its circum-centric dual. The smooth differential forms, in discrete setting, are mirrored by cochains on the primal and dual complexes, while the discrete exterior derivative is defined to be the coboundary operator. Discrete analogues of the Stokes-Dirac structure are the so-called simplicial Dirac structures defined on spaces of primal and dual discrete differential forms. These finite-dimensional Dirac structures offer a natural framework for the formulation of finite-dimensional port-Hamiltonian systems that emulate their infinite-dimensional counterparts. The resulting port-Hamiltonian systems are in the standard *input-output* form, unlike in (Golo, Talasila, van der Schaft, & Maschke, 2004), where the discretized models are *acausal* (given by a set of differential and algebraic equations). The explicit input-output form obtained by our scheme has the advantage from both numerical and control perspective over the implicit model presented in (Golo, Talasila, van der Schaft, & Maschke, 2004).

---

*Email addresses:* M.Seslija@rug.nl (Marko Seslija), J.M.A.Scherpen@rug.nl (Jacquélien M.A. Scherpen), A.J.van.der.Schaft@rug.nl (Arjan van der Schaft).

In this paper, we address the issue of matrix representations of simplicial Dirac structures by representing cochains by their coefficient vectors. In this manner, all linear operator from the continuous world can be represented by matrices, including the Hodge star, the coboundary and the trace operator. Firstly, we recall the definition of the Stokes-Dirac structure and port-Hamiltonian systems. In the third section, we define some essential concepts from discrete exterior calculus as developed in (Desbrun, Hirani, Leok, & Marsden, 2002), (Hirani, 2003). In order to allow the inclusion of nonzero boundary conditions on the dual cell complex, in (Seslija, van der Schaft, & Scherpen, 2011) we have adapted a definition of the dual boundary operator that leads to a discrete analogue of the integration by parts formula, which is a crucial ingredient in establishing simplicial Dirac structures on a primal simplicial complex and its circumcentric dual. We demonstrate how these simplicial Dirac structures relate to the spatially discretized wave equation on a bounded domain and to the telegraph equations on a segment. Towards the end of the paper, we consider the existence of structural invariants, which are crucial for the control by energy shaping.

*Note.* Some preliminary results of this paper are reported in (Seslija, Scherpen, & van der Schaft, 2012). The basic results of Section 3, 4 and 5 are already contained in (Seslija, van der Schaft, & Scherpen, 2011), with a main difference that the current presentation does not lean onto the heavy nomenclature of algebraic topology, but instead emphasizes matrix representations, making it more accessible and easier to implement.

## 2 Background of port-Hamiltonian systems

Dirac structures were originally developed in (Courant, 1990), (Dorfman, 1993) as a generalization of symplectic, presymplectic and Poisson structures. Later, Dirac structures were employed as the geometric formalism underpinning generalized interconnected and constrained Hamiltonian systems (van der Schaft, 2000), (van der Schaft & Maschke, 2002).

### 2.1 Dirac structures

Let  $\mathcal{Q}$  be a manifold and define a pairing on  $T\mathcal{Q} \oplus T^*\mathcal{Q}$  given by

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = e_1(f_2) + e_2(f_1).$$

For a subspace  $\mathcal{D}$  of  $T\mathcal{Q} \oplus T^*\mathcal{Q}$ , we define the orthogonal complement  $\mathcal{D}^\perp$  as the space of all  $(f_1, e_2)$  such that  $\langle\langle (f_1, e_2), (f_2, e_2) \rangle\rangle = 0$  for all  $(f_2, e_2)$ . A **Dirac structure** is then a subbundle  $\mathcal{D}$  of  $T\mathcal{Q} \oplus T^*\mathcal{Q}$  which satisfies  $\mathcal{D} = \mathcal{D}^\perp$ .

The notion of Dirac structures is suitable for the formulation of closed Hamiltonian systems, however, our

aim is a treatment of open Hamiltonian systems in such a way that some of the external variables remain free port variables. For that reason, let  $\mathcal{F}_b$  be a linear vector space of external flows, with the dual space  $\mathcal{F}_b^*$  of external efforts. We deal with Dirac structures on the product space  $\mathcal{Q} \times \mathcal{F}_b$ . The pairing on  $(T\mathcal{Q} \times \mathcal{F}_b) \oplus (T^*\mathcal{Q} \times \mathcal{F}_b^*)$  is given by

$$\begin{aligned} &\langle\langle ((f_1, f_{b,1}), (e_1, e_{b,1})), ((f_2, f_{b,2}), (e_2, e_{b,2})) \rangle\rangle \\ &= e_1(f_2) + e_{b,1}(f_{b,2}) + e_2(f_1) + e_{b,2}(f_{b,1}). \end{aligned} \quad (1)$$

A **generalized Dirac structure**  $\mathcal{D}$  is a subbundle of  $(T\mathcal{Q} \times \mathcal{F}_b) \oplus (T^*\mathcal{Q} \times \mathcal{F}_b^*)$  which is maximally isotropic under (1).

Consider a generalized Dirac structure  $\mathcal{D}$  on the product space  $\mathcal{Q} \times \mathcal{F}_b$ . Let  $H : \mathcal{Q} \rightarrow \mathbb{R}$  be a Hamiltonian. The **port-Hamiltonian system** corresponding to a 4-tuple  $(\mathcal{Q}, \mathcal{F}_b, \mathcal{D}, H)$  is defined by a set of smooth time-functions  $\{t \mapsto (x(t), f(t), e(t)) \in \mathcal{Q} \times \mathcal{F}_b \times \mathcal{F}_b^* | t \in I \subset \mathbb{R}\}$  satisfying the equation

$$(-\dot{x}(t), f(t), dH(x(t)), e(t)) \in \mathcal{D} \text{ for } t \in I. \quad (2)$$

The equation (2) implies the energy balance  $\frac{dH}{dt}(x(t)) = \langle dH(x(t)), \dot{x}(t) \rangle = \langle e(t), f(t) \rangle$ .

An important class of finite-dimensional port-Hamiltonian systems is given by

$$\begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)e \\ f &= g^\top(x) \frac{\partial H}{\partial x}, \end{aligned} \quad (3)$$

where for clarity we have omitted the argument  $t$ , and  $J : T^*\mathcal{Q} \rightarrow T\mathcal{Q}$  is a skew-symmetric vector bundle map and  $g : \mathcal{F}_b \rightarrow T\mathcal{Q}$  is the independent input vector field.

In this paper we deal exclusively with Dirac structures on linear spaces, which can be defined as follows. Let  $\mathcal{F}$  and  $\mathcal{E}$  be linear spaces. Given an  $f \in \mathcal{F}$  and an  $e \in \mathcal{E}$ , the pairing will be denoted by  $\langle e|f \rangle \in \mathbb{R}$ . By symmetrizing the pairing, we obtain a symmetric bilinear form  $\langle\langle, \rangle\rangle : \mathcal{F} \times \mathcal{E} \rightarrow \mathbb{R}$  naturally given as  $\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = \langle e_1|f_2 \rangle + \langle e_2|f_1 \rangle$ .

A **constant Dirac structure** is a linear subspace  $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$  such that  $\mathcal{D} = \mathcal{D}^\perp$ , with  $\perp$  standing for the orthogonal complement with respect to the bilinear form  $\langle\langle, \rangle\rangle$ .

### 2.2 Stokes-Dirac structure

The **Stokes-Dirac structure** is an infinite-dimensional Dirac structure that provides a foundation for the

port-Hamiltonian formulation of a class of distributed-parameter systems with boundary energy flow (van der Schaft & Maschke, 2002).

Hereafter, let  $M$  be an oriented  $n$ -dimensional smooth manifold with a smooth  $(n-1)$ -dimensional boundary  $\partial M$  endowed with the induced orientation, representing the space of spatial variables. By  $\Omega^k(M)$ ,  $k = 0, 1, \dots, n$ , denote the space of exterior  $k$ -forms on  $M$ , and by  $\Omega^k(\partial M)$ ,  $k = 0, 1, \dots, n-1$ , the space of  $k$ -forms on  $\partial M$ .

For any pair  $p, q$  of positive integers satisfying  $p + q = n + 1$ , define the flow and effort linear spaces by

$$\begin{aligned}\mathcal{F}_{p,q} &= \Omega^p(M) \times \Omega^q(M) \times \Omega^{n-p}(\partial M) \\ \mathcal{E}_{p,q} &= \Omega^{n-p}(M) \times \Omega^{n-q}(M) \times \Omega^{n-q}(\partial M).\end{aligned}$$

The bilinear form on the product space  $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$  is

$$\begin{aligned}& \langle\langle \underbrace{(f_p^1, f_q^1, f_b^1)}_{\in \mathcal{F}_{p,q}}, \underbrace{(e_p^1, e_q^1, e_b^1)}_{\in \mathcal{E}_{p,q}} \rangle\rangle, \underbrace{(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2)}_{\in \mathcal{E}_{p,q}} \rangle\rangle \\ &= \int_M e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_b^2 \wedge f_p^1 + e_q^2 \wedge f_q^1 \\ & \quad + \int_{\partial M} e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1.\end{aligned}\quad (4)$$

**Theorem 1 (van der Schaft & Maschke, 2002))**

Given linear spaces  $\mathcal{F}_{p,q}$  and  $\mathcal{E}_{p,q}$ , and the bilinear form  $\langle\langle, \rangle\rangle$ , define the following linear subspace  $\mathcal{D}$  of  $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$

$$\begin{aligned}\mathcal{D} &= \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid \right. \\ & \quad \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{pq+1}d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ & \quad \left. \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} \text{tr} & 0 \\ 0 & -(-1)^{n-q}\text{tr} \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} \right\},\end{aligned}\quad (5)$$

where  $d$  is the exterior derivative and  $\text{tr}$  stands for the trace operator on the boundary  $\partial M$ . Then  $\mathcal{D} = \mathcal{D}^\perp$ , that is,  $\mathcal{D}$  is a Dirac structure.

Consider a Hamiltonian density  $\mathcal{H} : \Omega^p(M) \times \Omega^q(M) \rightarrow \Omega^n(M)$  resulting in the Hamiltonian  $H = \int_M \mathcal{H} \in \mathbb{R}$ .

Setting the flows  $f_p = -\frac{\partial \alpha_p}{\partial t}$ ,  $f_q = -\frac{\partial \alpha_q}{\partial t}$  and the efforts  $e_p = \delta_p H$ ,  $e_q = \delta_q H$ , where  $(\delta_p H, \delta_q H) \in \Omega^{n-p}(M) \times \Omega^{n-q}(M)$  are the variational derivatives of  $H$  at  $(\alpha_p, \alpha_q)$ , the **distributed-parameter port-Hamiltonian system** is defined by the relation

$$\left( -\frac{\partial \alpha_p}{\partial t}, -\frac{\partial \alpha_q}{\partial t}, f_b, \delta_p H, \delta_q H, e_b \right) \in \mathcal{D}, \quad t \in \mathbb{R}. \quad (6)$$

Since  $\frac{dH}{dt} = \int_{\partial M} e_b \wedge f_b$ , the system is *lossless*.

### 3 Basics of discrete exterior calculus

In the discrete setting, the smooth manifold  $M$  is replaced by an oriented manifold-like simplicial complex. An  $n$ -dimensional **simplicial manifold**  $K$  is a simplicial triangulation of an  $n$ -dimensional polytope  $|K|$  with an  $(n-1)$ -dimensional boundary. Familiar examples of such discrete manifolds are meshes of triangles embedded in  $\mathbb{R}^3$  and tetrahedra obtained by tetrahedrization of 3-dimensional manifolds.

#### 3.1 Chains and cochains

The discrete analogue of a smooth  $k$ -form on the manifold  $M$  is a  $k$ -cochain on the simplicial complex  $K$ . A  $k$ -chain is a formal sum of  $k$ -simplices of  $K$  such that its value on a simplex changes sign when the simplex orientation is reversed. The free Abelian group generated by a basis consisting of oriented  $k$ -simplices with real-valued coefficients is  $C_k(K; \mathbb{R})$ . The space  $C_k(K; \mathbb{R})$  is a vector space with dimension equal to the number of  $k$ -simplices in  $K$ , which is denoted by  $N_k$ . The space of  $k$ -cochains is the vector space dual of  $C_k(K; \mathbb{R})$  denoted by  $C^k(K; \mathbb{R})$  or  $\Omega_d^k(K)$ , as a reminder that this is the **space of discrete  $k$ -forms**.

The **discrete exterior derivative** or the **coboundary operator**  $\mathbf{d}^k : \Omega_d^k(K) \rightarrow \Omega_d^{k+1}(K)$  is defined by duality to the boundary operator  $\partial_{k+1} : C_{k+1}(K; \mathbb{Z}) \rightarrow C_k(K; \mathbb{Z})$ , with respect to the natural pairing between discrete forms and chains. For a discrete form  $\alpha \in \Omega_d^k(K)$  and a chain  $c_{k+1} \in C_{k+1}(K; \mathbb{Z})$  we define  $\mathbf{d}^k$  by

$$\langle \mathbf{d}^k \alpha, c_{k+1} \rangle = \langle \alpha, (\mathbf{d}^k)^\top c_{k+1} \rangle = \langle \alpha, \partial_{k+1} c_{k+1} \rangle,$$

where the boundary operator  $\partial_{k+1}$  is the *incidence matrix* from the space of  $(k+1)$ -simplices to the space of  $k$ -simplices and is represented by a sparse  $N_{k+1} \times N_k$  matrix containing only 0 or  $\pm 1$  elements (Desbrun, Kanso, & Tong, 2008). The important property of the boundary operator is  $\partial_k \circ \partial_{k+1} = 0$ . The exterior derivative also satisfies  $\mathbf{d}^{k+1} \circ \mathbf{d}^k = 0$ , what is a discrete analogue of the vector calculus identities  $\text{curl} \circ \text{grad} = 0$  and  $\text{div} \circ \text{curl} = 0$ .

#### 3.2 Dual cell complex

An essential ingredient of discrete exterior calculus is the **dual complex** of a manifold-like simplicial complex. Given a simplicial complex  $K$ , we define its interior dual cell complex  $\star_1 K$  (block complex in terminology of algebraic topology (Munkres, 1984)) as a graph dual of  $K$  (gemoetrically) restricted to  $|K|$ . The boundary dual cell complex  $\star_b K$  is a dual to  $\partial K$ . The dual cell complex  $\star K$  is defined as  $\star K = \star_1 K \times \star_b K$ . A dual mesh  $\star_1 K$  is a dual to  $K$  in sense of a graph dual, and the dual of the boundary is equal to the boundary of the dual, that

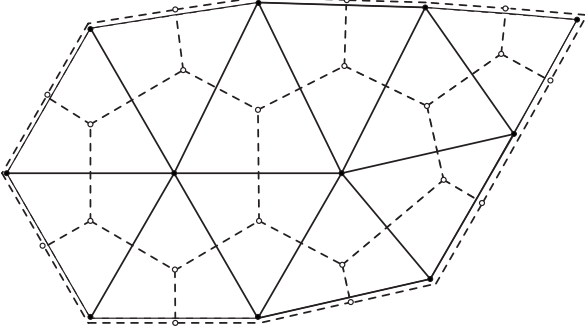


Fig. 1. A 2-dimensional simplicial complex  $K$  and its circumcentric dual cell complex  $\star K$  indicated by dashed lines. The boundary of  $\star K$  is the dual of the boundary of  $K$ .

is  $\partial(\star K) = \star(\partial K) = \star_b K$ . Because of duality, there is a one-to-one correspondence between  $k$ -simplices of  $K$  and interior  $(n - k)$ -cells of  $\star K$ . Likewise, to every  $k$ -simplex on  $\partial K$  there is a uniquely associated  $(n - 1 - k)$ -cell on  $\partial(\star K)$ . Fig. 1 illustrates the duality on a flat 2-dimensional simplicial complex.

Two most frequent dualities are the barycentric and circumcentric (Voronoi) dualities. An important property of the the Voronoi duality is that primal and dual cells are orthogonal to each other. This feature dramatically simplifies the discrete counterpart of the Hodge star, as will be shown in the next subsection. For this reason we shall be dealing with the circumcentric duality and require that the simplicial complex is well-centered (the circumcenters of all simplices of all dimensions lie in the interior of the corresponding simplices).

Everything that has been said about the primal discrete forms carries over to the dual cochains, which can be interpreted as covectors. The space of dual  $k$ -cochains will be denoted as  $\Omega_d^k(\star_i K)$ . The covectors will be labeled by a caret symbol, e.g.,  $\hat{\beta} \in \Omega_d^k(\star_i K)$ .

The **trace operator**  $\text{tr}^k : \Omega_d^k(K) \rightarrow \Omega_d^k(\partial K)$  is a matrix that isolates the members of a  $k$ -cochain vector assumed on the geometric boundary  $\partial K$ .

The **dual exterior derivative**  $\mathbf{d}_i^{n-k} : \Omega_d^{n-k}(\star_i K) \rightarrow \Omega_d^{n-k+1}(\star_i K)$  is defined by duality to the primal exterior operator  $\mathbf{d}^k$  as

$$\mathbf{d}_i^{n-k} = (-1)^k (\mathbf{d}^{k-1})^T.$$

The negative sign appears as the orientation of the dual is induced by the primal orientation.

The **dual boundary exterior derivative**  $\mathbf{d}_b^{n-k} : \Omega_d^{n-k}(\star_b K) \rightarrow \Omega_d^{n-k+1}(\star_b K)$  is defined as

$$\mathbf{d}_b^{n-k} = (-1)^{k-1} (\text{tr}^{k-1})^T.$$

For more details on geometric aspects of these operators confer to (Seslija, van der Schaft, & Scherpen, 2011) and for an example see Section 6.

### 3.3 Discrete wedge and Hodge operator

There exists a natural pairing, via the so-called primal-dual wedge product, between a primal  $k$ -cochain and a dual  $(n - k)$ -cochain. Let  $\alpha^k \in \Omega_d^k(K)$  and  $\hat{\beta}^{n-k} \in \Omega_d^{n-k}(\star_i K)$ . We define the discrete **primal-dual wedge product**  $\wedge : \Omega_d^k(K) \times \Omega_d^{n-k}(\star_i K) \rightarrow \mathbb{R}$  by  $\langle \alpha^k \wedge \hat{\beta}^{n-k}, V_{\sigma^k} \rangle = \langle \alpha^k, \sigma^k \rangle \langle \hat{\beta}^{n-k}, \star_i \sigma^k \rangle = (-1)^{k(n-k)} \langle \hat{\beta}^{n-k} \wedge \alpha^k, V_{\sigma^k} \rangle$ , where  $V_{\sigma^k}$  is the  $n$ -dimensional support volume obtained by taking the convex hull of the simplex  $\sigma^k$  and its dual  $\star_i \sigma^k$ .

The proposed definition of the dual boundary operator ensures the validity of the summation by parts relation that parallels the integration by parts formula for smooth differential forms. This summation by parts formula was presented in (Seslija, van der Schaft, & Scherpen, 2011) and its matrix version is given by the next proposition.

**Proposition 2** *Let  $K$  be an oriented well-centered simplicial complex. Given a primal  $(k - 1)$ -form  $\alpha$  and an internal dual  $(n - k)$ -discrete form  $\hat{\beta}_i \in \Omega_d^{n-k}(\star_i K)$  and a dual boundary form  $\hat{\beta}_b \in \Omega_d^{n-k}(\star_b K)$ , then*

$$\begin{aligned} \langle \mathbf{d}^{k-1} \alpha \wedge \hat{\beta}_i, K \rangle + (-1)^{k-1} \langle \alpha \wedge (\mathbf{d}_i^{n-k} \hat{\beta}_i + \mathbf{d}_b^{n-k} \hat{\beta}_b), K \rangle \\ = \langle \text{tr}^{k-1} \alpha \wedge \hat{\beta}_b, \partial K \rangle. \end{aligned}$$

**PROOF.** For completeness, we give this simple proof. Observe that

$$\begin{aligned} \langle \mathbf{d}^{k-1} \alpha \wedge \hat{\beta}_i, K \rangle &= (\mathbf{d}^{k-1} \alpha)^T \hat{\beta}_i = \alpha^t (\mathbf{d}^{k-1})^T \hat{\beta}_i \\ &= (-1)^k \alpha^T \mathbf{d}_i^{n-k} \hat{\beta}_i \\ &= (-1)^k \langle \alpha \wedge \mathbf{d}_i^{n-k} \hat{\beta}_i, K \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \alpha \wedge \mathbf{d}_b^{n-k} \hat{\beta}_b, K \rangle &= \alpha^T \mathbf{d}_b^{n-k} \hat{\beta}_b = ((\mathbf{d}_b^{n-k})^T \alpha)^T \hat{\beta}_b \\ &= (-1)^{k-1} (\text{tr}^{k-1} \alpha)^T \hat{\beta}_b \\ &= (-1)^{k-1} \langle \text{tr}^{k-1} \alpha \wedge \hat{\beta}_b, \partial K \rangle. \quad \square \end{aligned}$$

The support volumes of a simplex and its dual cell are the same, which suggests that there is a natural identification between primal  $k$ -cochains and dual  $(n - k)$ -cochains. In the exterior calculus for smooth manifolds, the Hodge star, denoted  $\ast_k$ , is an isomorphism between the space of  $k$ -forms and  $(n - k)$ -forms. The **discrete Hodge star** is a map  $\ast_k : \Omega_d^k(K) \rightarrow \Omega_d^{n-k}(\star_i K)$  defined by its value over simplices and their duals. In case of

the circumcentric duality, the Hodge star  $*_k$  is a diagonal  $N_k \times N_k$  matrix with the entry corresponding to a simplex  $\sigma^k$  being  $|\sigma^k|/|\star_i \sigma^k|$ .

Another possibility for the construction of the Hodge operator is to use Whitney forms. The Whitney map is an interpolation scheme for cochains. It maps discrete forms to square integrable forms that are piecewise smooth on each simplex. The Whitney maps are built from barycentric coordinate functions and the resulting matrix is sparse but in general *not* diagonal (Bossavit, 1998), (Hiptmair, 2002).

The linear operators used in this paper are succinctly presented in the following diagram

$$\begin{array}{ccccc}
\Omega^0(\partial K) & \xleftarrow{\text{tr}^0} & \Omega^0(K) & \xrightleftharpoons[*_0^{-1}]{*_0} & \Omega^n(\star_i K) \xleftarrow{\mathbf{d}_b^{n-1}} \Omega^{n-1}(\star_b K) \\
& & \downarrow \mathbf{d}^0 & & \uparrow \mathbf{d}_i^{n-1} \\
\Omega^1(\partial K) & \xleftarrow{\text{tr}^1} & \Omega^1(K) & \xrightleftharpoons[*_1^{-1}]{*_1} & \Omega^{n-1}(\star_i K) \xleftarrow{\mathbf{d}_b^{n-2}} \Omega^{n-2}(\star_b K) \\
& & \downarrow \mathbf{d}^1 & & \uparrow \mathbf{d}_i^{n-2} \\
& \vdots & \vdots & & \vdots \\
& & \downarrow \mathbf{d}^{n-2} & & \uparrow \mathbf{d}_i^1 \\
\Omega^{n-1}(\partial K) & \xleftarrow{\text{tr}^{n-1}} & \Omega^{n-1}(K) & \xrightleftharpoons[*_{n-1}^{-1}]{*_n} & \Omega^1(\star_i K) \xleftarrow{\mathbf{d}_b^0} \Omega^0(\star_b K) \\
& & \downarrow \mathbf{d}^{n-1} & & \uparrow \mathbf{d}_i^0 \\
& & \Omega^n(K) & \xrightleftharpoons[*_{n-1}^{-1}]{*_n} & \Omega^0(\star_i K)
\end{array}$$

#### 4 Simplicial Dirac structures

In this section, we develop the matrix representations of *simplicial Dirac structures*. These structures are discrete analogues of the Stokes-Dirac structure and as such are defined in terms of primal and dual cochains on the underlying discrete manifold (Seslija, van der Schaft, & Scherpen, 2011).

The role of the smooth manifold  $M$  in the discrete setting is played by an  $n$ -dimensional well-centered oriented manifold-like simplicial complex  $K$ . The flow and the effort spaces will be the spaces of complementary primal and dual forms. The elements of these two spaces are paired via the discrete primal-dual wedge product. Let

$$\begin{aligned}
\mathcal{F}_{p,q}^d &= \Omega_d^p(\star_i K) \times \Omega_d^q(K) \times \Omega_d^{n-p}(\partial(\star K)) \\
\mathcal{E}_{p,q}^d &= \Omega_d^{n-p}(K) \times \Omega_d^{n-q}(\star_i K) \times \Omega_d^{n-q}(\partial(\star K)).
\end{aligned}$$

The primal-dual wedge product ensures a *bijective* relation between the primal and dual forms, between the flows and efforts. A natural discrete mirror of the bilinear form (4) is a symmetric pairing on the product space  $\mathcal{F}_{p,q}^d \times \mathcal{E}_{p,q}^d$  defined by

$$\begin{aligned}
& \langle\langle \underbrace{(\hat{f}_p^1, f_q^1, f_b^1)}_{\in \mathcal{F}_{p,q}^d}, \underbrace{(e_p^1, \hat{e}_q^1, \hat{e}_b^1)}_{\in \mathcal{E}_{p,q}^d}, (\hat{f}_p^2, f_q^2, f_b^2, e_p^2, \hat{e}_q^2, \hat{e}_b^2) \rangle\rangle_d \\
&= \langle e_p^1 \wedge \hat{f}_p^2 + \hat{e}_q^1 \wedge f_q^2 + e_p^2 \wedge \hat{f}_p^1 + \hat{e}_q^2 \wedge f_q^1, K \rangle \\
&+ \langle \hat{e}_b^1 \wedge f_b^2 + \hat{e}_b^2 \wedge f_b^1, \partial K \rangle.
\end{aligned} \tag{7}$$

A discrete analogue of the Stokes-Dirac structure is the finite-dimensional Dirac structure constructed in Theorem 4.1 in (Seslija, van der Schaft, & Scherpen, 2011). The matrix representation of this Dirac structure is given by the following theorem.

**Theorem 3** *Given linear spaces  $\mathcal{F}_{p,q}^d$  and  $\mathcal{E}_{p,q}^d$ , and the bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle_d$ . The linear subspace  $\mathcal{D}_d \subset \mathcal{F}_{p,q}^d \times \mathcal{E}_{p,q}^d$  defined by*

$$\begin{aligned}
\mathcal{D}_d &= \{(\hat{f}_p, f_q, f_b, e_p, \hat{e}_q, \hat{e}_b) \in \mathcal{F}_{p,q}^d \times \mathcal{E}_{p,q}^d \mid \\
&\begin{bmatrix} \hat{f}_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r \mathbf{d}_i^{n-q} \\ \mathbf{d}^{n-p} & 0 \end{bmatrix} \begin{bmatrix} e_p \\ \hat{e}_q \end{bmatrix} + (-1)^r \begin{bmatrix} \mathbf{d}_b^{n-q} \\ 0 \end{bmatrix} \hat{e}_b, \\
&f_b = (-1)^p \text{tr}^{n-p} e_p\},
\end{aligned} \tag{8}$$

with  $r = pq + 1$ , is a Dirac structure with respect to the pairing  $\langle\langle \cdot, \cdot \rangle\rangle_d$ .

**PROOF.** Note that since  $\mathbf{d}_i^{n-q} = (-1)^q (\mathbf{d}^{n-p})^T$  and  $\mathbf{d}_b^{n-q} = (-1)^{n-p} (\text{tr}^{n-p})^T$ , the operator

$$\begin{bmatrix} 0 & (-1)^r \mathbf{d}_i^{n-q} & (-1)^r \mathbf{d}_b^{n-q} \\ \mathbf{d}^{n-p} & 0 & 0 \\ (-1)^p \text{tr}^{n-p} & 0 & 0 \end{bmatrix}$$

is skew-symmetric, and thus (8) is a *Poisson structure* on the state space  $\Omega_d^p(\star_i K) \times \Omega_d^q(K)$ .  $\square$

**Remark 1** *The Dirac structure (8) is purely topological and as such does not depend on the choice of the geometric duality. Thus, the equivalent result holds in case of the barycentric duality.*

The other discrete analogue of the Stokes-Dirac structure is defined on the spaces

$$\begin{aligned}
\tilde{\mathcal{F}}_{p,q}^d &= \Omega_d^p(K) \times \Omega_d^q(\star_i K) \times \Omega_d^{n-p}(\partial(\star K)) \\
\tilde{\mathcal{E}}_{p,q}^d &= \Omega_d^{n-p}(\star_i K) \times \Omega_d^{n-q}(K) \times \Omega_d^{n-q}(\partial(K)).
\end{aligned}$$

A natural discrete mirror of (4) in this case is a symmetric pairing defined by

$$\begin{aligned}
& \langle\langle \underbrace{(f_p^1, \hat{f}_q^1, \hat{f}_b^1)}_{\in \tilde{\mathcal{F}}_{p,q}^d}, \underbrace{(e_p^1, \hat{e}_q^1, e_b^1)}_{\in \tilde{\mathcal{E}}_{p,q}^d}, (f_p^2, \hat{f}_q^2, \hat{f}_b^2, e_p^2, \hat{e}_q^2, e_b^2) \rangle\rangle_{\tilde{d}} \\
&= \langle \hat{e}_p^1 \wedge f_p^2 + e_q^1 \wedge \hat{f}_q^2 + \hat{e}_p^2 \wedge f_p^1 + e_q^2 \wedge \hat{f}_q^1, K \rangle \\
&+ \langle \hat{e}_b^1 \wedge \hat{f}_b^2 + e_b^2 \wedge \hat{f}_b^1, \partial K \rangle.
\end{aligned}$$

**Theorem 4** The linear space  $\tilde{\mathcal{D}}_d$  defined by

$$\begin{aligned} \tilde{\mathcal{D}}_d = \{ & (f_p, \hat{f}_q, f_b, e_p, e_q, e_b) \in \tilde{\mathcal{F}}_{p,q}^d \times \tilde{\mathcal{E}}_{p,q}^d \mid \\ & \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{pq+1} \mathbf{d}^{n-q} \\ \mathbf{d}_i^{n-p} & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_p \\ e_q \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{d}_b^{n-p} \end{bmatrix} \hat{f}_b, \\ & e_b = (-1)^p \mathbf{tr}^{n-q} e_q \} \end{aligned} \quad (9)$$

is a Dirac structure with respect to the bilinear pairing  $\langle\langle, \rangle\rangle_{\tilde{d}}$ .

**PROOF.** The simplicial Dirac structure (9) is the dual of (8), and the proof is analogue to that of Theorem 3.  $\square$

In the following section, the simplicial Dirac structures (8) and (9) will be used as *terminus a quo* for the geometric formulation of spatially discrete port-Hamiltonian systems.

## 5 Port-Hamiltonian systems

Let a function  $\mathcal{H} : \Omega_d^p(\star_i K) \times \Omega_d^q(K) \rightarrow \mathbb{R}$  stand for the Hamiltonian  $(\hat{\alpha}_p, \alpha_q) \mapsto \mathcal{H}(\hat{\alpha}_p, \alpha_q)$ , with  $\hat{\alpha}_p \in \Omega_d^p(\star_i K)$  and  $\alpha_q \in \Omega_d^q(K)$ . A time derivative of  $\mathcal{H}$  along an arbitrary trajectory  $t \rightarrow (\hat{\alpha}_p(t), \alpha_q(t)) \in \Omega_d^p(\star_i K) \times \Omega_d^q(K)$ ,  $t \in \mathbb{R}$ , is

$$\frac{d}{dt} \mathcal{H}(\hat{\alpha}_p, \alpha_q) = \left\langle \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_p} \wedge \frac{\partial \hat{\alpha}_p}{\partial t} + \frac{\partial \mathcal{H}}{\partial \alpha_q} \wedge \frac{\partial \alpha_q}{\partial t}, K \right\rangle, \quad (10)$$

where the caret sign reminds that the quantity lives on the dual mesh. The relations between the simplicial-Dirac structure (8) and time derivatives of the variables are:  $\hat{f}_p = -\frac{\partial \hat{\alpha}_p}{\partial t}$ ,  $f_q = -\frac{\partial \alpha_q}{\partial t}$ , while the efforts are:  $e_p = \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_p}$ ,  $\hat{e}_q = \frac{\partial \mathcal{H}}{\partial \alpha_q}$ .

This allows us to define a time-continuous port-Hamiltonian system on a simplicial complex  $K$  (and its dual  $\star K$ ) by

$$\begin{aligned} \begin{bmatrix} -\frac{\partial \hat{\alpha}_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r \mathbf{d}_i^{n-q} \\ \mathbf{d}^{n-p} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_p} \\ \frac{\partial \mathcal{H}}{\partial \alpha_q} \end{bmatrix} + (-1)^r \begin{bmatrix} \mathbf{d}_b^{n-q} \\ 0 \end{bmatrix} \hat{e}_b, \\ f_b &= (-1)^p \mathbf{tr}^{n-p} \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_p}, \end{aligned} \quad (11)$$

where  $r = pq + 1$ .

The system (11) is evidently in the form (3). It immediately follows that  $\frac{d}{dt} \mathcal{H} = \langle \hat{e}_b \wedge f_b, \partial K \rangle$ , enunciating a fundamental property of the system: the increase in the energy on the domain  $|K|$  is equal to the power supplied to the system through the boundary  $\partial K$  and  $\partial(\star K)$ . The boundary efforts  $\hat{e}_b$  are the boundary control input and  $f_b$  are the outputs.

**Remark 2** Introducing a linear negative feedback control as  $\hat{e}_b = (-1)^{(n-p)(n-q)-1} \star_b f_b$ , where  $\star_b$  is the Hodge star on the boundary  $\partial K$ , leads to passivization of the lossless port-Hamiltonian system, i.e.,  $\frac{d}{dt} \mathcal{H} \leq -\langle f_b \wedge \star_b f_b, \partial K \rangle \leq 0$ . Furthermore, if the Hamiltonian is a  $\mathcal{K}_\infty$  function with a strict minimum that is a stationary set for the system (11), the equilibrium is asymptotically stable. A more elaborate control strategy can be the energy shaping method as is briefly discussed in Section 8.

An alternative formulation of a spatially discrete port-Hamiltonian system is given in terms of the simplicial Dirac structure (9). We start with the Hamiltonian function  $(\alpha_p, \hat{\alpha}_q) \mapsto \mathcal{H}(\alpha_p, \hat{\alpha}_q)$ , where  $\alpha_p \in \Omega_d^p(K)$  and  $\hat{\alpha}_q \in \Omega_d^q(\star_i K)$ . In a similar manner as in deriving (11), we introduce the input-output port-Hamiltonian system

$$\begin{aligned} \begin{bmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \hat{\alpha}_q}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r \mathbf{d}^{n-q} \\ \mathbf{d}_i^{n-p} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \alpha_p} \\ \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_q} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{d}_b^{n-p} \end{bmatrix} \hat{f}_b, \\ e_b &= (-1)^p \mathbf{tr}^{n-q} \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_q}. \end{aligned} \quad (12)$$

In contrast to (11), in the case of the formulation (12), the boundary flows  $\hat{f}_b$  can be considered to be freely chosen, while the boundary efforts  $e_b$  are determined by the dynamics. Note that the free boundary variables are *always* defined on the boundary of the dual cell complex.

## 6 Physical examples

In this section we consider the discrete wave equation on a 2-dimensional simplicial complex and the telegraph equations on a segment.

### 6.1 Two-dimensional wave equation

Consider the wave equation  $\mu \frac{\partial^2 u^c}{\partial t^2} = -E \Delta u^c$ , with  $u^c(t, z) \in \mathbb{R}$ ,  $z = (z_1, z_2) \in M$ , where  $\mu$  is the mass density,  $E$  is the Young's modulus,  $\Delta$  is the two-dimensional Laplace operator, and  $M$  is a compact surface with a closed boundary. Throughout, the superscript  $c$  designates the continuous quantities.

The energy variables are the 2-dimensional kinetic momentum  $p^c$ , and the 1-form elastic strain  $\epsilon^c$ . The coenergy variables are the 0-form velocity  $v^c$  and the 1-form stress  $\sigma^c$ . The energy density of the vibrating membrane is  $\mathcal{H}(p, \epsilon) = \frac{1}{2} (\epsilon^c \wedge \sigma^c + p^c \wedge v^c)$ , where the coenergy and energy variables are related by the constitutive relations  $\sigma^c = E \star \epsilon^c$  and  $v^c = 1/\mu \star p^c$ . The Hodge operator here corresponds to the standard Euclidean metric on  $M$ . The port-Hamiltonian formulation of the vibrating membrane in full details is given in (Golo, Talasila, van der Schaft, & Maschke, 2004).

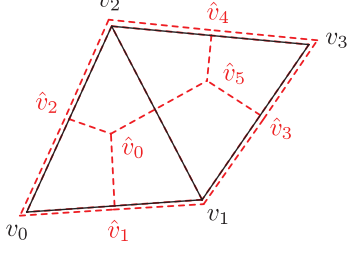


Fig. 2. A simplicial complex  $K$  consists of two triangles. The dual edges introduced by subdivision are shown dotted.

Let us now consider the simplicial Dirac structure underpinning the discretized two-dimensional wave equation. The energy variables of the discretized system are chosen as follows: the kinetic momentum is a dual 2-form whose time derivative is set to be  $\hat{f}_p$ , the elastic strain is a primal 1-form with time derivative corresponding to  $\hat{f}_q$ , the coenergy variables are a primal 0-form  $e_p$  and a dual 1-form  $\hat{e}_q$ . Such a formulation of the discrete wave equation is consonant with the simplicial Dirac structure (8) for the case when  $p = n = 2$  and  $q = 1$ , and is given by

$$\begin{bmatrix} \hat{f}_p \\ \hat{f}_q \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{d}_i^1 \\ \mathbf{d}^0 & 0 \end{bmatrix} \begin{bmatrix} e_p \\ \hat{e}_q \end{bmatrix} - \begin{bmatrix} \mathbf{d}_b^1 \\ 0 \end{bmatrix} \hat{e}_b,$$

$$f_b = \mathbf{tr}^0 e_p.$$

The boundary control variable is the 1-form stress  $\hat{e}_b$ , while the output is the boundary velocity. The Hamiltonian of the discrete model is

$$\mathcal{H} = \frac{1}{2} \left\langle \epsilon \wedge E * \epsilon + \hat{p} \wedge \frac{1}{\mu} *_0^{-1} \hat{p}, K \right\rangle.$$

The coenergy variables are the dual 1-form  $\hat{\sigma} = \frac{\partial \mathcal{H}}{\partial \epsilon} = E * \epsilon$  and the primal 0-form  $v = \frac{\partial \mathcal{H}}{\partial \hat{p}} = *_0^{-1} \hat{p}$ .

The resulting port-Hamiltonian system is

$$\begin{bmatrix} \frac{\partial \hat{p}}{\partial t} \\ \frac{\partial \epsilon}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{d}_i^1 \\ -\mathbf{d}^0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\mu} *_0^{-1} & 0 \\ 0 & E *_1 \end{bmatrix} \begin{bmatrix} \hat{p} \\ \epsilon \end{bmatrix} + \begin{bmatrix} \mathbf{d}_b^1 \\ 0 \end{bmatrix} \hat{e}_b$$

$$f_b = \frac{1}{\mu} \mathbf{tr}^0 *_0^{-1} \hat{p},$$

where the operators  $\mathbf{d}^0$ ,  $\mathbf{d}_i^1$ ,  $\mathbf{tr}^0 = (\mathbf{d}_b^1)^T$ ,  $*_1$ , and  $*_0^{-1}$  conform to the diagram at the end of Section 3 when  $n = 2$ .

**Example.** Consider a simplicial complex pictorially given by Fig. 2. The primal and dual 2-faces have counterclockwise orientations. The matrix representation of

the incidence operator  $\partial_1$ , from the primal edges to the primal vertices, is

	$[v_0, v_1]$	$[v_1, v_2]$	$[v_2, v_0]$	$[v_1, v_3]$	$[v_3, v_2]$
$v_0$	-1	0	0	0	0
$v_1$	1	-1	0	-1	0
$v_2$	0	1	-1	0	1
$v_3$	0	0	1	1	-1

while the discrete exterior derivative from the vertices to the edges is the transpose of the incidence operator, i.e.,  $\mathbf{d}^0 = \partial_1^T$ . The dual exterior derivative is  $\mathbf{d}_i^1 = -(\mathbf{d}^0)^T$ , while the matrix representation of the  $\mathbf{d}_b^1$  operator is

	$[\hat{v}_2, \hat{v}_1]$	$[\hat{v}_1, \hat{v}_3]$	$[\hat{v}_3, \hat{v}_4]$	$[\hat{v}_4, \hat{v}_2]$
$\star_1 v_0$	1	0	0	0
$\star_1 v_1$	0	1	0	0
$\star_1 v_2$	0	0	0	1
$\star_1 v_3$	0	0	1	0

The trace operator is  $\mathbf{tr}^0 = (\mathbf{d}_b^1)^T$ .

It is trivial to show

$$\begin{aligned} \langle \mathbf{d}^0 e_p \wedge \hat{e}_q, K \rangle + \langle e_p \wedge (\mathbf{d}_i^1 \hat{e}_q + \mathbf{d}_b^1 \hat{e}_b), K \rangle \\ = \hat{e}_b[\hat{v}_2, \hat{v}_1] f_b(v_0) + \hat{e}_b[\hat{v}_1, \hat{v}_3] f_b(v_1) \\ + \hat{e}_b[\hat{v}_3, \hat{v}_4] f_b(v_3) + \hat{e}_b[\hat{v}_4, \hat{v}_2] f_b(v_2), \end{aligned} \quad (13)$$

what confirms that the boundary terms genuinely live on the boundary of  $|K|$ .

If the geometric duality is circumcentric, the diagonal Hodge operators are

$$*_0 = \text{diag} \left( \frac{1}{|[*v_0]|}, \frac{1}{|[*v_1]|}, \frac{1}{|[*v_2]|}, \frac{1}{|[*v_3]|} \right)$$

$$*_1 = \text{diag} \left( \frac{|[v_0, v_1]|}{|[\hat{v}_1, \hat{v}_0]|}, \frac{|[v_1, v_2]|}{|[\hat{v}_0, \hat{v}_5]|}, \frac{|[v_2, v_0]|}{|[\hat{v}_2, \hat{v}_0]|}, \frac{|[v_1, v_3]|}{|[\hat{v}_3, \hat{v}_5]|}, \frac{|[v_3, v_2]|}{|[\hat{v}_4, \hat{v}_5]|} \right).$$

## 6.2 Telegraph equations

We consider an ideal lossless transmission line on a 1-dimensional simplicial complex. The energy variables are the charge density  $q \in \Omega_d^1(K)$ , and the flux density  $\hat{\phi} \in \Omega_d^1(\star K)$ , hence  $p = q = 1$ . The Hamiltonian representing the total energy stored in the transmission line with distributed capacitance  $C$  and distributed inductance  $\hat{L}$  is

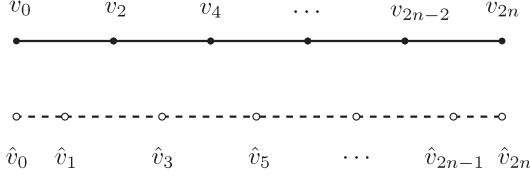


Fig. 3. The primal 1-dimensional simplicial complex  $K$ . By construction, the nodes  $\hat{v}_0$  and  $\hat{v}_{2n}$  are added to the boundary to insure that  $\partial(\star K) = \star(\partial K)$ .

$$\mathcal{H} = \left\langle \frac{1}{2C} q \wedge \star_1 q + \frac{1}{2\hat{L}} \hat{\phi} \wedge \star_0^{-1} \hat{\phi}, K \right\rangle, \quad (14)$$

where  $\star_0$  and  $\star_1$  are the discrete diagonal Hodge operators that relate the appropriate cochains according to the following schematic diagram

$$\begin{array}{ccccc} \Omega_d^0(\partial K) & \xleftarrow{\text{tr}^0} & \Omega_d^0(K) & \xrightarrow{\mathbf{d}^0} & \Omega_d^1(K) \\ \downarrow \star_b & & \downarrow \star_0 & & \downarrow \star_1 \\ \Omega_d^0(\partial(\star K)) & \xrightarrow{\mathbf{d}_b^0} & \Omega_d^1(\star_i K) & \xleftarrow{\mathbf{d}_i^0} & \Omega_d^0(\star_i K), \end{array}$$

where  $\star_b$  is the identity.

The co-energy variables are:  $\hat{e}_p = \frac{\partial \mathcal{H}}{\partial q} = \star \frac{q}{C} = \hat{V}$  representing voltages and  $e_q = \frac{\partial \mathcal{H}}{\partial \phi} = \star \frac{\phi}{L} = I$  currents.

Selecting  $f_p = -\frac{\partial q}{\partial t}$  and  $\hat{f}_q = -\frac{\partial \hat{\phi}}{\partial t}$  leads to the port-Hamiltonian formulation of the telegraph equations

$$\begin{bmatrix} -\frac{\partial q}{\partial t} \\ -\frac{\partial \hat{\phi}}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{d}^0 \\ \mathbf{d}_i^0 & 0 \end{bmatrix} \begin{bmatrix} \star_1 \frac{q}{C} \\ \star_0^{-1} \frac{\hat{\phi}}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{d}_b^0 \end{bmatrix} \hat{f}_b \quad (15)$$

$$e_b = -\text{tr}^0 \star_0^{-1} \frac{\hat{\phi}}{L},$$

where  $\hat{f}_b$  are the input voltages and  $e_b$  are the output currents.

In the case we want to have the electrical currents as the inputs, the charge and the flux densities would be defined on the dual mesh and the primal mesh, respectively. Instead of the port-Hamiltonian system in the form (15), the discretized telegraph equations would be in the form (11). The charge density is defined on the dual cell complex as  $\hat{q} \in \Omega_d^1(\star_i K)$  and the discrete flux density is  $\phi \in \Omega_d^1(K)$ . The finite-dimensional port-Hamiltonian system is of the form

$$\begin{bmatrix} -\frac{\partial \hat{q}}{\partial t} \\ -\frac{\partial \phi}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{d}_i^0 \\ \mathbf{d}^0 & 0 \end{bmatrix} \begin{bmatrix} \star_0^{-1} \frac{\hat{q}}{C} \\ \star_1 \frac{\phi}{L} \end{bmatrix} + \begin{bmatrix} \mathbf{d}_b^0 \\ 0 \end{bmatrix} \hat{e}_b \quad (16)$$

$$f_b = -\text{tr}^0 \star_0^{-1} \frac{\hat{q}}{C},$$

where  $\hat{e}_b$  are the input currents and  $f_b$  are the output voltages.

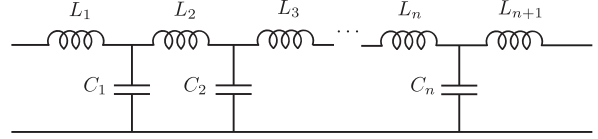


Fig. 4. The finite-dimensional approximation of the lossless transmission line when the inputs are voltages and the outputs are currents. The inductances  $L_1, \dots, L_{n+1}$  are the values that the discrete distributed inductance  $\hat{L}$  takes on the simplices  $[\hat{v}_0, \hat{v}_1], \dots, [\hat{v}_{2n-1}, \hat{v}_{2n}]$ ; the capacitances  $C_1, \dots, C_n$  are the values  $C$  takes on  $[v_0, v_2], \dots, [v_{2n-2}, v_{2n}]$ .

The exterior derivative  $\mathbf{d}^0 : \Omega_d^0(K) \rightarrow \Omega_d^1(K)$  is the transpose of the incidence matrix of the primal mesh. The discrete derivative  $\mathbf{d}_i^0 : \Omega_d^0(\star_i K) \rightarrow \Omega_d^1(\star_i K)$  in the matrix notation is the incidence matrix of the primal mesh. Thus, we have

$$(\mathbf{d}_i^0)^T = \mathbf{d}^0 = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, \quad (17)$$

$$\text{tr}^0 = (\mathbf{d}_b^0)^T = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \quad (18)$$

**Remark 3** The discrete analogue of the Stokes-Dirac structure obtained in (Golo, Talasila, van der Schaft, & Maschke, 2004) is a finite-dimensional Dirac structure, but not a Poisson structure. The implication of this on the physical realization is that, in contrast to our results, the transmission line in the finite-dimensional case is not only composed of inductors and capacitors but also of transformers.

The physical realizations of the port-Hamiltonian systems (15) and (16) are given on Fig. 4 and Fig. 5, respectively. Stabilization of either of those systems is easily achieved by terminating boundary ports with resistive elements, what is a practical application of the passivization explained in Remark 2.

The accuracy of the proposed method is  $1/n$  (see (Seslija, van der Schaft, & Scherpen, 2011)).

## 7 Conservation laws

Let us consider the existence of conservation laws and structural invariants for the port-Hamiltonian systems on simplicial complexes.

### 7.1 Finite-dimensional invariants

The following proposition gives the conditions for the existence of conservation laws in the discrete setting.



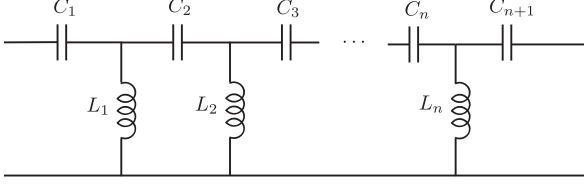


Fig. 5. The finite-dimensional approximation of the lossless transmission line when the inputs are currents and the outputs are voltages. The inductances are:  $L_1 = \int_{[v_0, v_2]} L^c = L([v_0, v_2])$ ,  $L_2 = \int_{[v_2, v_4]} L^c = L([v_2, v_4])$ , ...,  $L_n = \int_{[v_{2n-2}, v_{2n}]} L^c = L([v_{2n-2}, v_{2n}])$ ; the values of capacitors are:  $C_1 = \int_{[\hat{v}_0, \hat{v}_1]} C^c = \hat{C}([\hat{v}_0, \hat{v}_1])$ ,  $C_2 = \int_{[\hat{v}_1, \hat{v}_3]} C^c = \hat{C}([\hat{v}_1, \hat{v}_3])$ ,  $C_3 = \int_{[\hat{v}_3, \hat{v}_5]} C^c = \hat{C}([\hat{v}_3, \hat{v}_5])$ , ...,  $C_{n+1} = \int_{[\hat{v}_{2n-1}, \hat{v}_{2n}]} C^c = \hat{C}([\hat{v}_{2n-1}, \hat{v}_{2n}])$ .

**Proposition 5** Consider the port-Hamiltonian system (11). Let  $(\hat{\alpha}_p, \alpha_q) \mapsto C(\hat{\alpha}_p, \alpha_q)$  be a real-valued function. Then

$$\frac{\partial C}{\partial \hat{\alpha}_p} \in \ker \mathbf{d}^{n-p} \quad (19)$$

$$\frac{\partial \hat{C}}{\partial \alpha_q} \in \ker \mathbf{d}_i^{n-q}, \quad (20)$$

iff  $C$  is a conservation law for the port-Hamiltonian system (11) satisfying

$$\frac{dC}{dt} = (f_b^C)^T \hat{e}_b, \quad (21)$$

where  $f_b^C = -(-1)^{q(p+1)} \mathbf{tr}^{n-p} \frac{\partial C}{\partial \hat{\alpha}_p}$ .

**PROOF.** Differentiating  $C$  along the flow of the system (11), we have

$$\begin{aligned} \frac{dC}{dt} &= \left\langle \frac{\partial C}{\partial \hat{\alpha}_p} \wedge \frac{\partial \hat{\alpha}_p}{\partial t} + \frac{\partial C}{\partial \alpha_q} \wedge \frac{\partial \alpha_q}{\partial t}, K \right\rangle \\ &= (-1)^{pq} \frac{\partial^T C}{\partial \hat{\alpha}_p} \left( \mathbf{d}_i^{n-q} \frac{\partial \hat{\mathcal{H}}}{\partial \alpha_q} + \mathbf{d}_b^{n-q} \hat{e}_b \right) \\ &\quad - (-1)^{q(n-q)} \left( \mathbf{d}^{n-q} \frac{\partial \mathcal{H}}{\alpha_q} \right)^T \frac{\partial C}{\partial \alpha_q} \\ &= (-1)^{q(p+1)} \left( \mathbf{d}^{n-p} \frac{\partial C}{\partial \hat{\alpha}_p} \right)^T \frac{\partial \hat{\mathcal{H}}}{\partial \alpha_q} \\ &\quad - (-1)^{q(p+1)} \left( \mathbf{tr}^{n-p} \frac{\partial C}{\partial \hat{\alpha}_p} \right)^T \hat{e}_b \\ &\quad + (-1)^{pq} \left( \mathbf{d}_i^{n-q} \frac{\partial C}{\partial \alpha_q} \right)^T \frac{\partial \mathcal{H}}{\partial \hat{\alpha}_p}, \end{aligned} \quad (22)$$

where we have used the fact that  $\mathbf{d}_i^{n-q} = (-1)^q (\mathbf{d}^{n-p})^T$  and  $\mathbf{d}_b^{n-q} = (-1)^{n-p} (\mathbf{tr}^{n-p})^T$ . Furthermore, regardless

of  $\mathcal{H}$ , the result (21) follows iff (19) and (20) hold.  $\square$

**Remark 4** If either  $\hat{e}_b = 0$  or  $f_b^C = 0$ , the quantity  $C$  satisfying (19) and (20) is a conserved quantity—a Casimir function.

## 7.2 One-dimensional domain

An interesting case for which it is possible explicitly to solve (19) is when  $p = n$ . The matrix  $\mathbf{d}^0$  is nothing but the transpose of the incidence matrix  $\partial_1$ , from the set of edges to the set of vertices, on a connected graph. It is a well-known property of any incidence matrix  $\partial_1$  that  $\ker \partial_1^T = \text{span } \mathbf{1}$ , where  $\mathbf{1}$  stands for the vector with all elements equal 1. A direct consequence of this is that  $\frac{\partial C}{\partial \hat{\alpha}_p} = \mathbf{1}$  up to a multiplicative constant.

In the one-dimensional case the null space of  $\mathbf{d}_i^0$  is trivial, cf. (17), what allows us to explicitly express the conservation law.

**Corollary 6** Consider the port-Hamiltonian system (11), with  $p = q = n = 1$ , on a one-dimensional simplicial manifold given on Figure 3. The quantity  $C_p = \mathbf{1}^T \hat{\alpha}_p = \hat{\alpha}_p([\hat{v}_0, \hat{v}_1]) + \sum_{k=1}^{n-1} \hat{\alpha}_p([\hat{v}_{2k-1}, \hat{v}_{2k+1}]) + \hat{\alpha}_p([\hat{v}_{2n-1}, \hat{v}_{2n}])$  satisfies the balance law

$$\frac{dC_p}{dt} = \hat{e}_b(\hat{v}_0) - \hat{e}_b(\hat{v}_{2n}). \quad (23)$$

In case of the telegraph equations on the segment  $M = [0, 1]$ , the total charge  $C_q^c = \int_0^1 q^c(t, z) dz$  as well as the total magnetic flux  $C_\phi^c = \int_0^1 \phi^c(t, z) dz$  are both conservation laws. In the discrete setting, the *only* conservation law for the system (16) is the total charge  $C_q = \mathbf{1}^T \hat{q}$  whose derivative along the admissible trajectories is  $\frac{dC_q}{dt} = \hat{e}_b(\hat{v}_0) - \hat{e}_b(\hat{v}_{2n})$ . Similarly, the total flux  $C_\phi = \mathbf{1}^T \hat{\phi}$  in the system (15) satisfies the balance law  $\frac{dC_\phi}{dt} = \hat{f}_b(\hat{v}_0) - \hat{f}_b(\hat{v}_{2n})$ , where  $\hat{f}_b(\hat{v}_0)$  and  $\hat{f}_b(\hat{v}_{2n})$  are input currents. These result differ from those presented in (Macchelli, 2011), where both the total flux and total charge are conserved.

## 8 Energy-Casimir method

Consider the interconnection of (11) with the (possibly nonlinear) integrator

$$\frac{d\zeta}{dt} = \mathbf{g}_c u_c \quad (24)$$

$$y_c = \mathbf{g}_c^T \frac{\partial H_c}{\partial \zeta}, \quad (25)$$

where  $\zeta \in \mathbb{R}^m$ ,  $\mathbf{g}_c \in \mathbb{R}^{m \times N_b}$  with  $N_b = \dim \Omega_d^{n-q}(\partial(\star K))$ , input  $u_c$ , output  $y_c$ , and  $\zeta \mapsto H_c(\zeta)$  the controller's

Hamiltonian. The interconnection is power-preserving with  $u_c = f_b$  and  $e_b = -y_c$ . The composition is the port-Hamiltonian system in the form

$$\begin{bmatrix} \frac{\partial \hat{\alpha}_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \\ \frac{d\zeta}{dt} \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{r-1} \mathbf{d}_i^{n-q} & (-1)^r \mathbf{d}_b^{n-q} \mathbf{g}_c^T \\ \mathbf{d}^{n-p} & 0 & 0 \\ (-1)^p \mathbf{g}_c \mathbf{tr}^{n-p} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_{cl}}{\partial \hat{\alpha}_p} \\ \frac{\partial H_{cl}}{\partial \alpha_q} \\ \frac{\partial H_{cl}}{\partial \zeta} \end{bmatrix}, \quad (26)$$

with  $(\hat{\alpha}_p, \alpha_q, \zeta) \mapsto H_{cl}(\hat{\alpha}_p, \alpha_q, \zeta)$  is the closed-loop Hamiltonian  $H_{cl}(\hat{\alpha}_p, \alpha_q, \zeta) = \mathcal{H}(\hat{\alpha}_p, \alpha_q) + H_c(\zeta)$ .

The energy shaping for the system (26) is achieved by restricting the behavior of (26) to a certain subspace (van der Schaft, 2000). To this end, we look at the Casimir functions of the closed-loop system.

**Proposition 7** *The real-valued function  $(\hat{\alpha}_p, \alpha_q, \zeta) \mapsto C(\hat{\alpha}_p, \alpha_q, \zeta)$  is a Casimir function of the closed system (26) iff*

$$\begin{aligned} \frac{\partial C}{\partial \hat{\alpha}_p} &\in \ker \mathbf{d}^{n-p} \cap \ker (\mathbf{g}_c \mathbf{tr}^{n-p}) \\ \begin{bmatrix} \frac{\partial C}{\partial \alpha_q} \\ \frac{\partial C}{\partial \zeta} \end{bmatrix} &\in \ker \begin{bmatrix} \mathbf{d}_i^{n-q} & (-1)^{n-q} \mathbf{d}_b^{n-q} \mathbf{g}_c^T \end{bmatrix}. \end{aligned} \quad (27)$$

**PROOF.** Solving  $\frac{d}{dt} C(\hat{\alpha}_p, \alpha_q, \zeta) = 0$  irrespective of  $H_{cl}$  directly leads to (27).  $\square$

**Remark 5** *Since the structural matrix of the port-Hamiltonian system (11) is not of full rank in case when  $\mathbf{g}_c$  is identity, not all Casimirs of (26) are of the form  $C(\hat{\alpha}_q, \alpha_q, \zeta) = S_i(\hat{\alpha}_p, \alpha_q) - \zeta_i$ ,  $i = 1, \dots, m$ .*

**Remark 6** *In case when  $p = q = m = n = 1$  and  $\mathbf{g}_c = [1, 1]$  the only Casimir for the system (26) is  $\mathbf{1}^T \alpha_q + \zeta$ .*

## 9 Final remarks

An important avenue for the future research is to make a connection between continuous and discretized systems in presence of a port-Hamiltonian controller. Here, gauging the input-output errors is of a great significance for the robustness.

A possible interconnection of a distributed-parameter and a finite-dimensional port-Hamiltonian system can be established via the interface supplied by a simplicial triangulation of the spatial manifold. Energy shaping of the closed-loop system can be understood as a structural selection of the finite-dimensional controller. This will be explored in the forthcoming publication.

## References

- Bossavit, A. (1998). *Computational electromagnetism. Variational formulations, complementarity, edge elements*. Academic Press Inc., San Diego, CA.
- Courant, T. (1990). Dirac manifolds. *Trans. American Math. Soc.*, 319, pp. 631–661.
- Desbrun, M., Hirani, A.N., Leok, M., & Marsden, J.E. (2002). Discrete Exterior Calculus. arXiv:math/0508341v2
- Desbrun, M., Kanso, E., & Tong, Y. (2008). Discrete differential forms for computational modeling. *Discrete differential geometry*, Oberwolfach Seminars, Volume 38, Part IV, 287–324.
- Dorfman, I. (1993). *Dirac Structures and Integrability of Nonlinear Evolution Equations*. John Wiley, Chichester.
- Hirani, A. N. (2003). *Discrete exterior calculus*. Ph.D. thesis, California Institute of Technology.
- Golo, G., Talasila, V., van der Schaft, A.J., & Maschke, B. (2004). Hamiltonian discretization of boundary control systems. *Automatica*, vol. 40, no. 5, pp. 757–771, May 2004.
- Hiptmair, R. (2002). Finite elements in computational electromagnetism. In *Acta Numerica*, pp. 237–339, Cambridge University Press.
- Macchelli, A. (2011). Energyshaping of distributed parameter port-Hamiltonian systems based on finiteelement approximation. *Systems & Control Letters*, Volume 60, Issue 8, pp. 579–589.
- Munkres, J.R. (1984). *Elements of Algebraic Topology*, Addison-Wesley.
- van der Schaft, A.J. (2000). *L2-Gain and Passivity Techniques in Nonlinear Control*, Lect. Notes in Control and Information Sciences, Springer-Verlag, Berlin, p. xvi+249.
- van der Schaft, A.J., & Maschke, B.M. (2002). Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics*, vol. 42, pp. 166–194.
- Schöberl, M., & Schlacher, K. (2011). First-order Hamiltonian field theory and mechanics. *Math. Comput. Model. Dyn. Syst.* 17(1), 105–121.
- Seslija, M., van der Schaft, A.J., & Scherpen, J.M.A. (2011). Discrete Exterior Geometry Approach to Structure-Preserving Discretization of Distributed-Parameter Port-Hamiltonian Systems. *Journal of Geometry and Physics*, Volume 62, Issue 6, pp. 1509–153.
- Seslija, M., Scherpen, J.M.A., & van der Schaft, A.J. (2012). Port-Hamiltonian systems on discrete manifolds. *MathMod 2012 – 7th Vienna International Conference on Mathematical Modelling*, Vienna.